

A note on rainbow matchings in properly edge-coloured graphs *

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Abstract

A rainbow matching in an edge-coloured graph is a matching such that its edges have distinct colours. We show that every properly edge-coloured graph G with $|G| \geq (9\delta(G) - 5)/2$ has a rainbow matching of size $\delta(G)$, improving a result of Diemunsch et al.

1 Introduction

Let $G = (V, E)$ be a simple undirected graph without loops. Write $|G|$, $\delta(G)$ and $\Delta(G)$ for the order, minimum degree and maximum degree of G respectively. A *proper edge-colouring* of G is a function $c : E \rightarrow \{1, 2, \dots\}$ such that any two adjacent edges have distinct colours. If G is assigned such a colouring c , then we say that G is a *properly edge-coloured graph*. Denote the colour of the edge $e \in E$ by $c(e)$. A subgraph H of G is *rainbow* if its edges have distinct colours. The study of rainbow matchings began with a conjecture of Ryser [5], which states that every Latin square of odd order contains a Latin transversal. An equivalent statement is that for n odd, every n -edge-coloured of complete bipartite graph $K_{n,n}$ contains a rainbow perfect matching. A survey on rainbow matchings and other rainbow subgraphs in edge-coloured graphs appears in [3].

LeSaulnier et al. [4] proved that if G is a properly edge-coloured graph with $G \neq K_4$ or $|G| \neq \delta(G) + 2$, then G contains a rainbow matching of size $\lceil \delta(G)/2 \rceil$. If we further impose that $|G| \geq 8\delta(G)/5$, then Wang [6] showed that G contains a rainbow matching of size $\lfloor 3\delta(G)/5 \rfloor$. In the same paper, Wang asked whether there exists a function $f(n)$ such that every properly edge-coloured graph G with $|G| \geq f(\delta(G))$ contains a rainbow matching of size $\delta(G)$. Clearly, if $f(n)$ exists, then $f(n) \geq 2n$. In fact, $f(n) > 2n$ for n even as there exist $n \times n$ Latin square that have no Latin transversal (see [1] and [7]). Diemunsch et al. [2] gave an affirmative answer to Wang's question and showed that $f(n) = \lfloor 13n/2 - 23/2 + 41/(8n) \rfloor + 1$ suffices. In this article, we show that $f(n) = (9n - 5)/2$ would also be sufficient, improving the values of $f(n)$ for $n \geq 5$.

Theorem 1.1. *Every properly edge-coloured graph G with $|G| \geq (9\delta(G) - 5)/2$ has a rainbow matching of size $\delta(G)$.*

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2 Proof of Theorem 1.1

Let G be a properly edge-coloured graph with minimum degree δ and $n = |G| \geq (9\delta - 5)/2$ vertices. The theorem trivially holds for $\delta = 1$, so we may assume that $\delta \geq 2$. Suppose the theorem is false. Let G be a counterexample with δ minimal, so $n \geq (9\delta - 5)/2$ and G does not contain a rainbow matching of size δ . We break down the proof into a series of simple claims.

Claim 2.1. $\Delta(G) \leq 3(\delta - 1)$.

Proof. Let v be a vertex in G such that $d(v) > 3(\delta - 1)$. By the minimal counterexample, there is a rainbow matching M of size $\delta - 1$ in $G \setminus \{v\}$. Recall that G is properly edge-coloured and $d(v) > 3(\delta - 1)$, so there exists a vertex $u \in V(G) \setminus (V(M) \cup \{v\})$ such that the colour $c(uv)$ does not appear in M . Thus, $M \cup \{uv\}$ is a rainbow matching of size δ , a contradiction. \square

Let a be the size of the largest monochromatic matching M_0 in G . First we show that $a \geq 2$.

Claim 2.2. $a \geq 2$.

Proof. If $a = 1$, then G is rainbow. Thus, it is enough to show that G contains a matching of size δ . Let $M = \{x_i y_i : 1 \leq i \leq m\}$ be a matching of maximal size. If $m \geq \delta$, then we are done. For each $v \notin V(M)$, the neighbourhood of v must lie in $V(M)$. Since $\delta(G) \geq \delta$, there exists an integer $1 \leq i \leq m$ such that both $x_i v$ and $y_i v$ are edges. As $|G| \geq 3\delta$, there exist an integer $1 \leq i \leq t$ and vertices $v, v' \notin V(M)$ such that $v \neq v'$ and both $x_i v$ and $y_i v'$ are edges. Thus, $M \cup \{x_i v, y_i v'\} \setminus \{x_i y_i\}$ is a matching of size $m + 1$ contradicting the maximality of M . \square

Fix a monochromatic matching M_0 of size a . By the minimal counterexample, there exists a rainbow matching $M = \{x_i y_i : 1 \leq i \leq \delta - 1\}$ of size $\delta - 1$ in $G - M_0$. Without loss of generality, we may assume that $c(x_i y_i) = i$ for $1 \leq i \leq \delta - 1$ and M_0 is of colour δ . Note that every edge in M_0 intersects with $V(M)$ or else we can enlarge M to a rainbow matching of size δ . Set $W = V(G) \setminus V(M)$, so $|W| = n - 2(\delta - 1)$. We say that an edge uv is *good* if its colour is not in $\{1, \dots, \delta - 1\}$ and one of its vertices is in W . If there exists a good edge uv in $G[W]$ (the induced edge-coloured subgraph of G on W), then $M \cup \{uv\}$ is a rainbow matching of size δ . Thus, we may assume that every good edge is incident with $V(M)$, so every good edge lies between $V(M)$ and W .

Claim 2.3. For $1 \leq i \leq \delta - 1$, if x_i is incident with at least three good edges, then no good edge is incident with y_i , and vice versa.

Proof. Suppose the contrary, so x_i is incident with at least three good edges and $y_i u$ is a good edge. Since x_i is incident with at least three good edges, there exists $w \in W$ such that $c(x_i w) \neq c(y_i u)$ and $u \neq w$. Then $M \cup \{x_i w, y_i u\} \setminus \{x_i y_i\}$ is a rainbow matching of size δ , a contradiction. \square

A vertex $v \in V(M)$ is *good* if v is incident with at least seven good edges. By Claim 2.3, we may assume without loss of generality that $\{x_1, \dots, x_r\}$ is the set of good vertices. Let $W' = W \cup \{y_1, \dots, y_r\}$.

Claim 2.4. No edge uv in $G[W']$ has colour in $\{1, \dots, r\}$.

Proof. Suppose the contrary, so we may assume that there is an edge uv in $G[W']$ such that $c(uv) = 1$. Since G is properly edge-coloured, $v \neq y_1 \neq u$. If $u, v \in W$, then there exists a vertex $w \in W$ such that x_1w is good and $u \neq w \neq v$ and so $M \cup \{uv, x_1w\} \setminus \{x_1y_1\}$ is a rainbow matching of size δ , a contradiction. Next assume that $u \in W$ and $v \notin W$, so $r \geq 2$ and $v = y_2$ say. There exist distinct vertices $w_1, w_2 \in W \setminus \{u\}$ such that x_1w_1 and x_2w_2 are good edges with $c(x_1w_1) \neq c(x_2w_2)$. Then, $M \cup \{y_2u, x_1w_1, x_2w_2\} \setminus \{x_1y_1, x_2y_2\}$ is a rainbow matching of size δ , a contradiction. Finally, assume that $u, v \in W$ so $r \geq 3$ and $u = y_2$ and $v = y_3$ say. Since x_1, x_2 and x_3 are good, there exist distinct vertices $w_1, w_2, w_3 \in W$ such that x_1w_1, x_2w_2 and x_3w_3 are good edges with distinct colours. Thus, $M \cup \{y_2y_3, x_1w_1, x_2w_2, x_3w_3\} \setminus \{x_1y_1, x_2y_2, x_3y_3\}$ is a rainbow matching of size δ , a contradiction. \square

We say that an edge uv is *nice* if its colour is not in $\{r+1, \dots, \delta-1\}$ and one of its vertices is in W' . Note that every good edge is nice. Recall that every good edge is incident with $V(M)$. By Claim 2.3 and Claim 2.4, no nice edge lies in $G[W']$. Hence, every nice edge lies between W' and $V(G) \setminus W'$. A vertex $v \in V(M) \setminus \{x_1, \dots, x_r, y_1, \dots, y_r\}$ is *nice* if v is incident with at least seven nice edges. Note that if there is no good vertex i.e. $r = 0$, the definition of good and nice vertex are the same and so there is also no nice vertex. Next, we show the analogue of Claim 2.3 and Claim 2.4 for nice vertices and edges.

Claim 2.5. *For $r+1 \leq i \leq \delta-1$, if x_i is incident with at least three nice edges, then no nice edge is incident with y_i , and vice versa.*

Proof. Suppose the contrary, so x_i is incident with at least three nice edges and y_iu is a nice edge for some $r+1 \leq i \leq \delta-1$. Here, we only consider one particular case as each remaining case can be verified using a similar argument. Suppose that $r \geq 4$, $u = y_1$ and $c(y_iy_1) = 2$. Since x_i is nice, there exists a vertex $v \in W'$ such that x_iv is nice and $v \neq u = y_1$ and $c(x_iv) \neq c(y_iy_1) = 2$. Assume that $v = y_3$ and $c(x_iy_3) = 4$. Recall that x_1, \dots, x_4 are good vertices, so each is joined to at least seven good edges. Thus, there exist distinct vertices $w_1, w_2, w_3, w_4 \in W$ such that x_1w_1, x_2w_2, x_3w_3 and x_4w_4 are good edges with distinct colours. Therefore, $M \cup \{y_iy_1, x_iy_3, x_1w_1, x_2w_2, x_3w_3, x_4w_4\} \setminus \{x_1y_1, x_2y_2, x_3y_3, x_4y_4, x_iy_i\}$ is a rainbow matching of size δ , a contradiction. \square

By Claim 2.5, we may assume that $\{x_{r+1}, x_{r+2}, \dots, x_{r+s}\}$ is the set of nice vertices.

Claim 2.6. *No edge uv in $G[W']$ has colour in $\{1, \dots, r+s\}$.*

Proof of claim. By Claim 2.4, the claim holds if $s = 0$. Assume that $s \geq 1$, so $r \geq 1$. Suppose that there is an edge uv in $G[W']$ with $c(uv) = r+1$ say. Again, we only consider one particular case as each remaining case can be verified using a similar argument. Suppose that $r \geq 4$, $u = y_1$ and $v = y_2$. Since x_{r+1} is nice, there exists a vertex $v' \in W'$ such that $v'x_{r+1}$ is a nice edge. Assume that $v = y_3$ and $c(x_{r+1}y_3) = 4$. Recall that x_1, \dots, x_4 are good vertices, so each is joined to at least seven good edges. Thus, there exist distinct vertices $w_1, w_2, w_3, w_4 \in W$ such that x_1w_1, x_2w_2, x_3w_3 and x_4w_4 are good edges with distinct colours. Therefore, $M \cup \{y_1y_2, x_{r+1}y_3, x_1w_1, x_2w_2, x_3w_3, x_4w_4\} \setminus \{x_1y_1, x_2y_2, x_3y_3, x_4y_4, x_{r+1}y_{r+1}\}$ is a rainbow matching of size δ , a contradiction. \square

Next, we count the number of nice edges in G' .

Claim 2.7. *There are at most $(3\delta - 9 + s)r + 6(\delta - 1)$ nice edges.*

Proof. Recall that $V \setminus W' = \{x_1, \dots, x_{\delta-1}, y_{r+1}, \dots, y_{\delta-1}\}$ and every nice edge lies between W' and $V \setminus W'$. For $1 \leq i \leq r$, x_i is joined to at most $3(\delta-1)$ nice edges as $\Delta(G) \leq 3(\delta-1)$. By Claim 2.5 and the definition of nice, for $r+1 \leq i \leq r+s$ there are at most $r+6$ nice edges joining to x_i and none to y_i . For $r+s+1 \leq i \leq \delta-1$, there are at most six nice edges joining to x_i or y_i by Claim 2.5. Therefore, the number of nice edges is at most

$$3(\delta-1)r + (r+6)s + 6(\delta-1-r-s) = (3\delta-9+s)r + 6(\delta-1).$$

□

Recall that $V(M)$ is incident with a edges of colour δ . Hence, there are at least $2(a-\delta+1)$ vertices $v \in V(M)$ such that $c(vw) = \delta$ for some $w \in W$. Let t be the number of integers i such that $r+s+1 \leq i \leq \delta-1$, and $c(x_iw) = \delta$ or $c(y_iw) = \delta$ for some $w \in W$. Without loss of generality, we may assume that $r+s+1, \dots, r+s+t$ are such i . By Claim 2.3 and Claim 2.5, we have

$$t \geq a - \delta + 1 - \frac{r+s}{2} \text{ and } r+s+t \leq \delta-1. \quad (1)$$

Claim 2.8. *For $r+s+1 \leq i \leq r+s+t$, there are at most one edge of colours i in $G[W]$.*

Proof of claim. Suppose uv and $u'v'$ are edges of colour i in $G[W]$ for some $r+s+1 \leq i \leq r+s+t$. Without loss of generality, we may assume that there exists $w \in W$ such that $c(x_iw) = \delta$ and $u \neq w \neq v$. Hence, $M \cup \{uv, x_iw\} \setminus \{x_iy_i\}$ is a rainbow matching of size δ , a contradiction. □

Now, we count the number of nice edges from W' to $V \setminus W'$. Recall that there are at most a edges of the same colour. By Claim 2.6, there is no edge in $G[W']$ of colour $r+1 \leq i \leq r+s$. Thus, for $r+1 \leq i \leq r+s$ there are at most $a-1$ vertices in W' that are incident with an edge of colour i . By Claim 2.8, for $r+s+1 \leq i \leq r+s+t$, there are at most a vertices in W that is incident with an edge of colour i . Recall that $W' \setminus W = \{y_1, \dots, y_r\}$. For $r+s+1 \leq i \leq r+s+t$, there are at most $a+r$ vertices in W' that are incident with an edge of colour i . For $r+s+t+1 \leq i \leq \delta-1$, there are at most $2(a-1)$ vertices in W' that is incident with an edge of colour i . Thus, the number of nice edges from W' to $V \setminus W'$ is at least

$$\begin{aligned} & \delta|W'| - (a-1)s - (a+r)t - 2(a-1)(\delta-1-r-s-t) \\ &= \delta n - 2\delta(\delta-1) - (a-1)(2\delta-2-2r-s) + (a-2)t + (\delta-t)r \\ &\geq \delta n - 2\delta(\delta-1) - (a-1)(2\delta-2-2r-s) + (a-2)t + (r+s+1)r, \end{aligned}$$

where we recall that $|W'| = |W| + r = n - 2(\delta-1) + r$ and (1). Since there are at most $(3\delta-9+s)r + 6(\delta-1)$ nice edges in G by Claim 2.7,

$$\delta n \leq (3\delta-10-r)r - (a-2)t + 2(\delta+3)(\delta-1) + (a-1)(2\delta-2-2r-s). \quad (2)$$

For the remaining of the proof, we bound the right hand side of the above inequality from above to obtain a contradiction. Note that the coefficient of t is $-(a-2) \leq 0$ by Claim 2.2, so we can take the minimum value of t . By (1), $t \geq a - \delta + 1 - (r+s)/2$.

If $a \leq \delta-1 + (r+s)/2$, then we take $t = 0$. The coefficient of a becomes $2\delta-2-2r-s \geq 2(\delta-1-r-s) \geq 0$. Thus, by taking $a = \delta-1 + (r+s)/2$, (2) becomes

$$\delta n \leq 2(2\delta+1)(\delta-1) + (2\delta-7-2r)r - (3r+s-2)s/2.$$

Recall that if $s \geq 1$, then $r \geq 1$. Hence, $(3r + s - 2)s \geq 0$ and so

$$\delta n \leq 2(2\delta + 1)(\delta - 1) + (2\delta - 7 - 2r)r \leq 9\delta^2/2 - 11\delta/2 + 33/8$$

by taking $r = \delta/2 - 7/4$. Hence, $n \leq 9\delta/2 - 11/2 + 33/8\delta$, a contradiction.

If $a \geq \delta - 1 + (r + s)/2$, we take $t = a - \delta + 1 - (r + s)/2 \geq 0$. Then, (2) becomes

$$\delta|G| \leq (3\delta - 1 - (3r + s)/2 - a)a + (3\delta - 9 - r)r + 2(\delta + 1)(\delta - 1) \quad (3)$$

If $(3\delta - 1)/2 - (3r + s)/4 \leq \delta - 1 + (r + s)/2$, then right hand side is maximum when $a = \delta - 1 + (r + s)/2$, which corresponds to the case when $a \leq \delta - 1 + (r + s)/2$ and so we are done. Hence, we may assume that $(3\delta - 1)/2 - (3r + s)/4 > \delta - 1 + (r + s)/2$, so

$$5r + 3s < 2(\delta + 1). \quad (4)$$

Now we take $a = (3\delta - 1)/2 - (3r + s)/4$, so

$$\begin{aligned} \delta n &\leq (3\delta - 1 - (3r + s)/2)^2/4 + (3\delta - 7 + s)r + 2\delta(\delta - 1) \\ &= \left(-\frac{7r}{16} + \frac{3\delta}{4} - \frac{33}{4}\right)r + \left(\frac{3r}{8} + \frac{s}{16} - \frac{3\delta}{4} + \frac{1}{4}\right)s + \frac{17\delta^2}{4} - \frac{3\delta}{2} - \frac{7}{4} \\ &\leq \left(-\frac{7r}{16} + \frac{3\delta}{4} - \frac{33}{4}\right)r + \left(\frac{3(5r + 3s)}{40} - \frac{3\delta}{4} + \frac{1}{4}\right)s + \frac{17\delta^2}{4} - \frac{3\delta}{2} - \frac{7}{4} \\ &\leq \left(-\frac{7r}{16} + \frac{3\delta}{4} - \frac{33}{4}\right)r + \frac{(2 - 3\delta)s}{5} + \frac{17\delta^2}{4} - \frac{3\delta}{2} - \frac{7}{4} \\ &\leq \left(-\frac{7r}{16} + \frac{3\delta}{4} - \frac{33}{4}\right)r + \frac{17\delta^2}{4} - \frac{3\delta}{2} - \frac{7}{4}. \end{aligned}$$

Note that there is a maximal point at $r = 6(d - 11)/7$. Recall (4) that $r \leq 2(\delta + 2)/5$. Therefore,

$$n \leq \begin{cases} (17\delta - 6)/4 - 7/(4\delta) & \text{if } \delta \leq 11 \\ (32\delta - 60)/7 + 260/(7\delta) & \text{if } 12 \leq \delta \leq 22 \\ 112(\delta - 1)/25 - 863/(100\delta) & \text{if } \delta \geq 23 \end{cases}$$

by taking $r = 0$, $r = 6(d - 11)/7$ and $r = 2(\delta + 2)/5$ respectively. Moreover, $n < (9\delta - 5)/2$ a contradiction. This completes the proof of Theorem 1.1.

References

- [1] R. A. Brualdi and H. J. Ryser, *Combinatorial matrix theory*, Encyclopedia of Mathematics and its Applications, vol. 39, Cambridge University Press, 1991.
- [2] J. Diemunsch, M. Ferrara, C. Moffatt, F. Pfender, and P.S. Wenger, *Rainbow matchings of size $\delta(G)$ in properly edge-colored graphs*, Arxiv preprint arXiv:1108.2521 (2011).
- [3] M. Kano and X. Li, *Monochromatic and heterochromatic subgraphs in edge-colored graphs—a survey*, Graphs Combin. **24** (2008), 237–263.
- [4] T. D. LeSaulnier, C. Stocker, P. S. Wenger, and D. B. West, *Rainbow matching in edge-colored graphs*, Electron. J. Combin. **17** (2010), no. 1, Note 26, 5.

- [5] H. J. Ryser, *Neuere probleme der kombinatorik*, Vortrage über Kombinatorik Oberwolfach, Mathematisches Forschungsinstitut Oberwolfach (1967), 24–29.
- [6] G. Wang, *Rainbow matchings in properly edge colored graphs*, Electron. J. Combin. **18** (2011), no. 1, Note 162, 7.
- [7] I. M. Wanless, *Transversals in latin squares: a survey*, Surveys in Combinatorics 2011, London Math. Soc., 2011.